

The size of the largest component below phase transition in inhomogeneous random graphs.

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Abstract

We study the "rank 1 case" of the inhomogeneous random graph model. In the subcritical case we derive an exact formula for the asymptotic size of the largest connected component scaled to $\log n$. This result is new, it completes the corresponding known result in the supercritical case. We provide some examples of application of a new formula.

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1 Introduction.

1.1 Inhomogeneous random graphs.

We consider here a subclass of a general inhomogeneous random graph model $G^{\mathcal{V}}(n, \kappa)$ with a vertex space

$$\mathcal{V} = (S, \mu, (x_1, \dots, x_n)_{n \geq 1})$$

introduced in [1]. Here S is a separable metric space and μ is a Borel probability measure on S . Recall the basic definitions and assumptions from [1]. For each n the set of vertices of the graph $G^{\mathcal{V}}(n, \kappa)$ is a deterministic or random sequence x_1, \dots, x_n of points in S , such that for any μ -continuity set $A \subseteq S$

$$\frac{\#\{i : x_i \in A\}}{n} \xrightarrow{P} \mu(A). \quad (1.1)$$

Given the sequence x_1, \dots, x_n , we let $G^{\mathcal{V}}(n, \kappa)$ be the random graph on these vertices, such that any two vertices x_i and x_j are connected by an edge independently of the others and with a probability

$$p_{x_i, x_j}(n) = \min\{\kappa_n(x_i, x_j)/n, 1\}, \quad (1.2)$$

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where κ_n is a symmetric nonnegative measurable function on $S \times S$. We assume also that for all $x(n) \rightarrow x$ and $y(n) \rightarrow y$ in S

$$\kappa_n(x(n), y(n)) \xrightarrow{a.s.} \kappa(x, y) \quad (1.3)$$

as $n \rightarrow \infty$, where the kernel κ is *graphical* on \mathcal{V} , which means that

- (i) κ is continuous *a.s.* on $S \times S$;
- (ii) $\kappa \in L^1(S \times S, \mu \times \mu)$;
- (iii)

$$\frac{1}{n} \mathbf{E} e(G^\mathcal{V}(n, \kappa)) \rightarrow \frac{1}{2} \int_{S^2} \kappa(x, y) d\mu(x) d\mu(y),$$

where $e(G)$ denotes the number of edges in a graph G .

It was observed in [6] that random graphs can be naturally related to a certain branching process underlying the algorithm of revealing a connected component in a graph. This idea was extended in [7] for some inhomogeneous graph model, where a multi-type branching process was introduced to study the connectivity of the graph. But it was not until [1] that a comprehensive theory of inhomogeneous random graphs was developed, which provided a unified approach to many models studied previously.

Already in [6] it was shown that in the classical random graph model $G_{n,p}$ with $p = c/n$ the size of the largest connected component scaled to n asymptotically equals the survival probability of the associated branching process. A correspondent result was established in [1] for a general model $G^\mathcal{V}(n, \kappa)$ described above. We shall recall this result here. Let $C_1(G)$ denote the size (the number of vertices) of the largest connected component in a graph G . Then Theorem 3.1 from [1] states that

$$\frac{C_1(G^\mathcal{V}(n, \kappa))}{n} \xrightarrow{P} \rho_\kappa := \int_S \rho_\kappa(x) d\mu(x), \quad (1.4)$$

where $\rho_\kappa(x)$ is the survival probability of a multi-type Galton-Watson process $B_\kappa(x)$ defined as follows. The type space of $B_\kappa(x)$ is S , and initially there is a single particle of type $x \in S$. Then at any step, a particle of type $x \in S$ is replaced in the next generation by a set of particles where the number of particles of type y has a Poisson distribution with intensity $\kappa(x, y) d\mu(y)$. It was also proved in [1] that $\rho_\kappa(x)$ is the maximum solution to

$$\rho_\kappa(x) = 1 - e^{-\int_S \kappa(x, y) \rho_\kappa(y) d\mu(y)}.$$

Whether ρ_κ is zero or strictly positive depends only on the norm of an integral operator T_κ defined as

$$(T_\kappa f)(x) = \int_S \kappa(x, y) f(y) d\mu(y) \quad (1.5)$$

with norm

$$\|T_\kappa\| = \sup\{\|T_\kappa f\|_2 : f \geq 0, \|f\|_2 \leq 1\}.$$

Then according to Theorem 3.1 from [1] the survival probability

$$\rho_\kappa \begin{cases} > 0, & \text{if } \|T_\kappa\| > 1, \\ = 0, & \text{if } \|T_\kappa\| \leq 1. \end{cases} \quad (1.6)$$

Hence, while (1.4) describes rather accurate the size of the largest connected component above the phase transition, i.e., when $\|T_\kappa\| > 1$, all what we can get from (1.4) when $\|T_\kappa\| \leq 1$ is $C_1(G^\mathcal{V}(n, \kappa)) = o_P(n)$. Only under an additional assumption

$$\sup_{x, y, n} \kappa_n(x, y) < \infty \quad (1.7)$$

Theorem 3.12 in [1] proves in the case $\|T_\kappa\| < 1$ that $G^\mathcal{V}(n, \kappa) = O(\log n)$ **whp** (which means "with high probability", i.e., with probability tending to one as $n \rightarrow \infty$).

In the case of a homogeneous random graph $G_{n,p}$ with $p = c/n$ the following convergence in probability (and even more precise result) was derived already in [3]: if $c < 1$ then

$$\frac{C_1(G_{n,c/n})}{\log n} \xrightarrow{P} \frac{1}{c - 1 + |\log c|} \quad (1.8)$$

as $n \rightarrow \infty$. However, the method used in [3] is not applicable for an inhomogeneous model.

1.2 Main results.

Our aim here is to derive the asymptotics of the size of the largest component scaled to $\log n$ (similar to (1.8)) for inhomogeneous random graph model in the case $\|T_\kappa\| < 1$. We show, that this is also directly related to the parameters of the introduced branching process B_κ .

Assume from now on that $S \subseteq \mathbf{R}_+$ is finite or countable, μ is a probability on S , and a graphical kernel κ on $S \times S$ has a form

$$\kappa(x, y) = c\psi(x)\psi(y), \quad (1.9)$$

where ψ is a positive function on S and c is a positive constant. We consider a graph $G^\mathcal{V}(n, \kappa)$ on the vertex space \mathcal{V} which satisfies condition (1.1), and given x_1, \dots, x_n , the edges are independent and have probabilities (1.2) with $\kappa_n = \kappa$, i.e.,

$$p_{x_i, x_j}(n) = \min \left\{ \frac{c\psi(x_i)\psi(x_j)}{n}, 1 \right\}.$$

In this case operator T_κ defined in (1.5) has rank 1 (giving the name "the rank 1 case" of inhomogeneous random graph model, see Chapter 16.4 in [1]), and

$$\|T_\kappa\| = c \sum_S \psi^2(x) \mu(x). \quad (1.10)$$

Assumption 1.1. *Let function ψ satisfy one of the following conditions: either*

$$\sup_{x \in S} \psi(x) < \infty, \quad (1.11)$$

or for some monotone increasing unbounded function ψ_0 and positive constants $A_1 \leq A_2$

$$A_1 \psi_0(x) \leq \psi(x) \leq A_2 \psi_0(x), \quad (1.12)$$

for all large x , and

$$\sum_S e^{a\psi(x)} \mu(x) < \infty \quad (1.13)$$

for some positive a .

Also we shall assume that for any $\varepsilon > 0$ and $q > 0$

$$\mathbf{P} \left\{ \left| \frac{\#\{i : x_i = k\}}{n} - \mu(k) \right| \leq \varepsilon e^{q\psi(k)} \mu(k), \quad k \in S \right\} \rightarrow 1 \quad (1.14)$$

as $n \rightarrow \infty$. Notice that when S is finite, convergence (1.14) trivially follows by (1.1).

Let $B_\kappa(x)$ be a branching process defined as above: it starts with one particle of type $x \in S$, and then at any step, a particle of type $x \in S$ produces $Po(\kappa(x, y) \mu(y))$ number of offspring of each type $y \in S$. Denote $\mathcal{X}(x)$ the size of the total progeny of $B_\kappa(x)$, and let

$$r(c) = \sup \{z \geq 1 : \sum_S \mu(x) \psi(x) \mathbf{E} z^{\mathcal{X}(x)} < \infty\}. \quad (1.15)$$

Theorem 1.1. *Let $\kappa(x, y) = c\psi(x)\psi(y)$ and set*

$$c^{cr} := \left(\sum_S \psi^2(x) \mu(x) \right)^{-1}. \quad (1.16)$$

Under Assumption 1.1 and (1.14) we have

$$\frac{C_1(G^\nu(n, \kappa))}{\log n} \xrightarrow{P} \frac{1}{\log r(c)} \quad (1.17)$$

as $n \rightarrow \infty$, where

$$r(c) \begin{cases} > 1, & \text{if } c < c^{cr}, \\ = 1, & \text{if } c \geq c^{cr}. \end{cases} \quad (1.18)$$

Observe that due to (1.10) one has

$$\|T_\kappa\| < 1 \quad \Leftrightarrow \quad c < c^{cr}. \quad (1.19)$$

Hence, the statement of Theorem 1.1 is exactly complementary to (1.4) (under the conditions of Theorem 1.1), since

$$r(c) > 1 \text{ implies } \rho_\kappa = 0,$$

as well as $\rho_\kappa > 0$ implies $r(c) = 1$. Notice, however, that when $c = c^{cr}$ then both $r(c) = 1$ and $\rho_\kappa = 0$, and none of statements (1.4) or (1.17) provides substantial information.

It is rather obvious that one can extend Theorem 1.1 for the case of non-countable S under similar assumptions, replacing sum by the integral with respect to μ . It is less apparent, but one may conjecture as well, that the statement similar to (1.17) is not restricted to the rank 1 case only.

The rank 1 case is proved to be versatile for applications. One may interpret $\psi(x)$ as an "activity" of a vertex of type x . One particularly often seen choice of ψ is $\psi(x) = x$ on $S = \{1, 2, \dots\}$. Here "type x " can represent a degree of a node as in [2] or a size of a macro-vertex as in [9] (see also Chapter 16.4 in [1] on other examples). A special feature of the rank 1 case is that it allows one to compute $r(c)$ in a rather closed form as we state below.

Theorem 1.2. *Assume, the conditions of Theorem 1.1 are satisfied. Let X be a random variable in S with probability function μ . There exists a unique $y > 1$ which satisfies*

$$y = \frac{1}{c\mathbf{E}\psi(X)} \frac{\mathbf{E}\psi(X) \exp\{c\psi(X) (\mathbf{E}\psi(X)) (y-1)\}}{\mathbf{E}\psi^2(X) \exp\{c\psi(X) (\mathbf{E}\psi(X)) (y-1)\}}. \quad (1.20)$$

Then

$$r(c) = \frac{1}{c\mathbf{E}\psi^2(X) \exp\{c\psi(X) (\mathbf{E}\psi(X)) (y-1)\}}. \quad (1.21)$$

Notice that Theorems 1.1 and 1.2 immediately yield (1.8). Indeed, in the case of a homogeneous model $G_{n,c/n}$ we have $\psi(X) \equiv 1$ in Theorem 1.2, trivially implying $y = 1/c$, which together with (1.21) and (1.17) gives (1.8).

The result (1.17) is new for the inhomogeneous random graphs. (In fact, even in the case of $G_{n,c/n}$ the role of $r(c)$ was not disclosed previously.) In the subcritical case of $G_{n,c/n}$

the method of branching processes was used first in [6] to get a rough upper bound for the largest connected component. Then in [8] it was shown that for the same case one can get an optimal upper bound known from (1.8), again using the branching processes.

Recent study [9] indicated that an upper bound found there for the largest component in the subcritical case of some inhomogeneous random graph model should be the optimal one. We shall show here that the conjecture from [9] is a simple corollary of the following modification of Theorem 1.1.

Let L denote a component of a graph $G^\nu(n, \kappa)$. In some applications one studies a function of L in the form $\sum_{x_i \in L} \psi(x_i)$, which is natural to call an "activity of a component" if we call $\psi(x)$ an activity of a vertex of type x . A similar characteristic of a graph, called a "volume" was also treated in [4].

Consider again the branching processes $B_\kappa(x)$ defined above, which starts with one particle of type $x \in S$. Denote $\tilde{\mathcal{X}}(x)$ the set of all the offspring of the branching processes $B_\kappa(x)$. (Recall that previously we set $|\tilde{\mathcal{X}}(x)| = \mathcal{X}(x)$.) Let $\Phi(x)$ be the sum of "activities" of all the offspring of the branching processes $B_\kappa(x)$:

$$\Phi(x) = \sum_{v \in \tilde{\mathcal{X}}(x)} \psi(v).$$

This implies that $\Phi(x)$ satisfies the following equality in distribution

$$\Phi(x) \stackrel{d}{=} \psi(x) + \sum_{y \in S} \sum_{i=1}^{N_x(y)} \Phi_i(y), \quad (1.22)$$

where $N_x(y) \in \text{Po}(c\psi(x)\psi(y)\mu(y))$, independent for different x and y ; random variables $\Phi(x)$ and $\Phi_i(x), i \geq 1$, are *i.i.d.*, and also independent for different values of x ; and a sum over empty set is assumed to be zero. Define also similar to (1.15)

$$\alpha(c) = \sup \{z \geq 1 : \sum_S \psi(x) \mathbf{E} z^{\Phi(x)} \mu(x) < \infty\}. \quad (1.23)$$

Now we are ready to state another result similar to Theorem 1.1.

Theorem 1.3. *Let \mathcal{L} denote a set of all connected components in $G^\nu(n, \kappa)$. Under the conditions of Theorem 1.1 we have*

$$\frac{\max_{L \in \mathcal{L}} \sum_{x_i \in L} \psi(x_i)}{\log n} \xrightarrow{P} \frac{1}{\log \alpha(c)} \quad (1.24)$$

as $n \rightarrow \infty$, where

$$\alpha(c) \begin{cases} > 1, & \text{if } c < c^{cr}, \\ = 1, & \text{if } c \geq c^{cr}. \end{cases} \quad (1.25)$$

One can also find formula for $\alpha(c)$ similar to the one in Theorem 1.2.

1.3 Example.

Let $G_N(p, c)$ be a graph with the set of vertices $B(N) := \{-N, \dots, N\}^d$ in Z^d , $d \geq 1$, with two types of edges: the short-range edges connect independently with probability p each pair u and v if $|u - v| = 1$, and the long-range edges connect independently any pair of two vertices with probability $c/|B(N)|$. By this definition there can be none, one or two edges between two vertices in graph $G_N(p, c)$, and in the last case the edges are of different types. Assume, that $0 \leq p < p_c(d)$, where $p_c(d)$ is the percolation threshold in dimension d . As it is shown in [9], this graph is naturally related to the described above rank 1 case. Consider the subgraph of $G_N(p, c)$ induced by the short-range edges only, which is a purely bond percolation model. Let K_N denote the number of open clusters (i.e., connected by the short-range edges only), and let $\mathbf{X} = \{X_1, X_2, \dots, X_{K_N}\}$ denote the collection of all open clusters $X_i \subseteq B(N)$. Let also C denote an open cluster containing the origin. Recall that

$$\frac{K_N}{|B(N)|} \rightarrow \mathbf{E} \frac{1}{|C|} \quad (1.26)$$

a.s. and in L^1 as $N \rightarrow \infty$ (see, e.g., [5]). Call each set X_i a macro-vertex of type $|X_i|$.

Now given a collection of clusters \mathbf{X} , introduce another graph $\tilde{G}_N(\mathbf{X}, p, c)$, whose vertices are macro-vertices X_1, X_2, \dots, X_{K_N} . The probability that two (macro-)vertices X_i and X_j with $|X_i| = x$ and $|X_j| = y$ are connected is derived from the original model $G_N(p, c)$, which is

$$\tilde{p}_{xy}(N) = 1 - \left(1 - \frac{c}{|B(N)|}\right)^{xy} =: \frac{\kappa_{K_N}(x, y)}{K_N}. \quad (1.27)$$

Clearly, the size of the largest connected component in $G_N(p, c)$ has the following representation

$$C_1(G_N(p, c)) = \max_L \sum_{X_i \in L} |X_i| \quad (1.28)$$

where the maximum is taken over all connected components L in $\tilde{G}_N(\mathbf{X}, p, c)$.

It was shown in [9] that graph $\tilde{G}_N(\mathbf{X}, p, c)$ fits the definition of an inhomogeneous random graph. In particular,

$$\kappa_{K_N}(x, y) \xrightarrow{a.s.} \kappa(x, y) := c \mathbf{E}(|C|^{-1})xy \quad (1.29)$$

as $N \rightarrow \infty$, and

$$\frac{\#\{1 \leq i \leq K_N : |X_i| = k\}}{K_N} \xrightarrow{a.s.} \frac{1}{\mathbf{E}(|C|^{-1})} \frac{\mathbf{P}\{|C| = k\}}{k} =: \mu(k) \quad (1.30)$$

as $N \rightarrow \infty$. (We refer to [9] for the details.) It follows also from the results of [9], that model $\tilde{G}_N(\mathbf{X}, p, c)$ satisfies the conditions of Theorem 1.1 with $S = \{1, 2, \dots\}$, $\kappa(x, y) = c\mathbf{E}(|C|^{-1})xy$, (we set here $\psi(x) = x$), and μ defined in (1.30). In this case according to (1.5)

$$\|T_\kappa\| = c\mathbf{E}(|C|^{-1}) \sum_S \psi^2(x) \mu(x) = \sum_S x^2 \frac{1}{\mathbf{E}(|C|^{-1})} \frac{\mathbf{P}\{|C| = x\}}{x} = c\mathbf{E}|C|,$$

implying that $\|T_\kappa\| < 1$ if and only if $c < \frac{1}{\mathbf{E}|C|}$. Hence, taking into account (1.28) and also convergence (1.26) we readily get the result on $G_N(p, c)$ model.

Corollary 1.1. *Assume, that $d \geq 1$ and $0 \leq p < p_c(d)$. Then*

$$\frac{C_1(G_N(p, c))}{\log |B(N)|} \xrightarrow{P} \frac{1}{\log \gamma(p, c)}$$

as $N \rightarrow \infty$, where $\gamma(p, c) = \alpha(c\mathbf{E}(|C|^{-1}))$ and $\alpha(c)$ is defined by (1.23) with $\psi(x) = x$, and

$$\gamma(p, c) \begin{cases} > 1, & \text{if } c < \frac{1}{\mathbf{E}|C|}, \\ = 1, & \text{if } c \geq \frac{1}{\mathbf{E}|C|}. \end{cases} \quad (1.31)$$

□

This result was conjectured in [9], where it was proved that for any $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \mathbf{P} \left\{ \frac{C_1(G_N(p, c))}{\log |B(N)|} > \frac{1}{\log \gamma(p, c)} + \varepsilon \right\} = 0.$$

We shall also refer to [9] on more exact description of $\gamma(p, c)$ which is similar to the derivation of (1.21).

2 Proofs.

2.1 The generating function for the progeny of a branching process.

Recall that $\mathcal{X}(x)$ denote the total number of the particles (including the initial one) produced by the branching process $B_\kappa(x)$, and $\Phi(x)$ is the total activity as defined in (1.22). Let for

$$z \geq 1$$

$$h_z(x) = \mathbf{E}z^{\mathcal{X}(x)}, \quad g_z(x) = \mathbf{E}z^{\Phi(x)}.$$

Define also

$$H_z = \sum_S \psi(x) \mathbf{E}z^{\mathcal{X}(x)} \mu(x).$$

Then we rewrite (1.15)

$$r(c) = \sup\{z \geq 1 : H_z < \infty\}.$$

First we shall prove the following lemma, which in particular yields (1.18) and (1.25).

Lemma 2.1. *Let μ be a probability on S , and let function ψ be positive on S and satisfy (1.13). Write (as in (1.16))*

$$c^{cr} := \left(\sum_S \psi^2(x) \mu(x) \right)^{-1}.$$

Then

(I)

$$r(c) \begin{cases} > 1, & \text{if } c < c^{cr}, \\ = 1, & \text{if } c \geq c^{cr}; \end{cases} \quad (2.32)$$

(II)

$$\alpha(c) \begin{cases} > 1, & \text{if } c < c^{cr}, \\ = 1, & \text{if } c \geq c^{cr}; \end{cases} \quad (2.33)$$

(III) for all $n \geq 1$

$$\sup\{z \geq 1 : h_z(n) < \infty\} = r(c), \quad (2.34)$$

and

$$\sup\{z \geq 1 : g_z(n) < \infty\} = \alpha(c). \quad (2.35)$$

Proof. Note that function $h_z(k)$ (as a generating function for a branching process) satisfies the following equation

$$\begin{aligned} h_z(k) &= z \exp \left\{ \sum_{x \in S} \kappa(k, x) \mu(x) (h_z(x) - 1) \right\} \\ &= z \exp \left\{ c \psi(k) \sum_{x \in S} \psi(x) \mu(x) (h_z(x) - 1) \right\}. \end{aligned}$$

Let X denote a random variable in S with distribution μ . Then we can rewrite the last formula as follows

$$h_z(k) = z \exp \{c\psi(k)(H_z - \mathbf{E}\psi(X))\}. \quad (2.36)$$

Multiplying both sides by $\psi(k)\mu(k)$ and summing up over k we find for all $z < r(c)$

$$\begin{aligned} H_z &= \sum_{k \in S} \psi(k)\mu(k)z \exp \{c\psi(k)(H_z - \mathbf{E}\psi(X))\} \\ &= z\mathbf{E}\psi(X) \exp \{c\psi(X)(H_z - \mathbf{E}\psi(X))\}. \end{aligned} \quad (2.37)$$

Notice, that

$$H_1 = \mathbf{E}\psi(X),$$

and clearly, H_z is an increasing function of z . Hence, equation (2.37) has a finite solution for some $z > 1$ if and only if the equation

$$y = \frac{1}{\mathbf{E}\psi(X)} z \mathbf{E}\psi(X) \exp \{c\psi(X)(\mathbf{E}\psi(X))(y - 1)\}. \quad (2.38)$$

(for the same value z) has a root $y > 1$. It is easy to see taking into account assumption (1.13), that at least for some $y > 1$ and $z > 1$ function on the right in (2.38)

$$f(y, z) := z f(y) := z \frac{1}{\mathbf{E}\psi(X)} \mathbf{E}\psi(X) \exp \{c\psi(X)(\mathbf{E}\psi(X))(y - 1)\} \quad (2.39)$$

is increasing in both variables, it has all the derivatives of the second order, and $\frac{\partial^2}{\partial y^2} f(y, z) > 0$. Compute now

$$\frac{\partial}{\partial y} f(y, z)|_{y=1, z=1} = c\mathbf{E}\psi^2(X) = \frac{c}{c^{cr}}. \quad (2.40)$$

Hence, if $c \geq c^{cr}$ then for any $z > 1$ there is no solution $y \geq 1$ to (2.38). On the other hand, if $c < c^{cr}$ then there exists $z_0 > 1$ such that for all $1 \leq z \leq z_0$ there is a finite solution $y \geq 1$ to (2.38) which in turn implies existence of $z > 1$ for which (2.37) has a finite solution H_z . This proves (2.32). The statement (2.34) follows immediately by (2.36).

Exploring formula (1.22) it is easy to derive that function $g_z(k)$ satisfies the following equation

$$g_z(k) = z^{\psi(k)} \exp \left\{ c\psi(k) \sum_{x \in S} \psi(x)\mu(x) (g_z(x) - 1) \right\}.$$

Then with a help of random variable X we can rewrite the last formula as follows

$$g_z(k) = z^{\psi(k)} \exp \{c\psi(k)(G_z - \mathbf{E}\psi(X))\}, \quad (2.41)$$

where

$$G_z = \sum_S \psi(x) \mu(x) g_z(x).$$

Multiplying both sides of (2.41) by $\psi(k) \mu(k)$ and summing up over k we find for all $z < \alpha(c)$

$$\begin{aligned} G_z &= \sum_{k \in S} \psi(k) \mu(k) z^{\psi(k)} \exp \{c\psi(k)(G_z - \mathbf{E}\psi(X))\} \\ &= \mathbf{E} z^{\psi(X)} \psi(X) \exp \{c\psi(X)(G_z - \mathbf{E}\psi(X))\}. \end{aligned} \quad (2.42)$$

The rest of the proof of (2.33) and (2.35) is identical to that of (2.32) and (2.34). \square

2.2 Proof of Theorem 1.2.

We shall find $r(c) = z_0$ as the (unique!) value for which function y is tangent to $f(y, z_0)$ (see (2.39) and (2.38)) if $y \geq 1$.

Assume, for some $z_0 > 1$ the function y is tangent to $f(y, z)$, and let y_0 be the tangency point. Hence, z_0 and y_0 satisfy the following equations

$$z_0 f'(y_0) = 1,$$

$$z_0 f(y_0) = y_0,$$

which implies that y_0 is the unique solution to

$$y_0 = \frac{f(y_0)}{f'(y_0)} = \frac{1}{c \mathbf{E}\psi(X)} \frac{\mathbf{E}\psi(X) \exp \{c\psi(X) (\mathbf{E}\psi(X)) (y_0 - 1)\}}{\mathbf{E}\psi^2(X) \exp \{c\psi(X) (\mathbf{E}\psi(X)) (y_0 - 1)\}},$$

and then

$$r(c) = z_0 = \frac{1}{f'(y_0)} = \frac{1}{c \mathbf{E}\psi^2(X) \exp \{c\psi(X) (\mathbf{E}\psi(X)) (y_0 - 1)\}}. \quad (2.43)$$

This proves formula (1.21). \square

2.3 Proof of Theorem 1.1.

We shall assume here that ψ satisfies (1.12) and (1.13). In the case of (1.11) when $\inf_{x \in S} \psi(x) > 0$ one may set $\psi_0(x) = \text{const}$ in (1.12), and the proof will follow by the same argument. When (1.11) holds with $\inf_{x \in S} \psi(x) = 0$, it is easy to construct an upper and a lower approximations for the kernel κ (consult also [1] on approximations), so that the proof will again be reduced to the previous case.

2.3.1 The lower bound.

First we shall prove that for any $\delta > 0$

$$\mathbf{P} \left\{ C_1 \left(G^\vee(n, \kappa) \right) > \left(\frac{1}{\log r(c)} + \delta \right) \log n \right\} \rightarrow 0 \quad (2.44)$$

as $N \rightarrow \infty$.

Recall the usual algorithm of finding a connected component in a random graph. Conditionally on the set of vertices $\mathbf{V} := \{x_1, \dots, x_n\}$, take any vertex $x_i \in \mathbf{V}$ to be the root. Find all the vertices $\{v^1, v^2, \dots, v^m\}$ connected to this vertex x_i in the graph $G^\vee(n, \kappa)$, and then mark x_i as "saturated". Then for each non-saturated but already revealed vertex, we find all the vertices connected to it but which have not been used previously. We continue this process until we end up with a tree of saturated vertices.

Denote $\tau_n(x_i)$ the set of the vertices in the tree constructed according to the above algorithm with the root at a vertex x_i .

First we shall prove the following intermediate result.

Lemma 2.2. *If $c < c^{cr}$ then*

$$\lim_{n \rightarrow 0} \mathbf{P} \left\{ C_1 \left(G^\vee(n, \kappa) \right) > n^{1/2} \right\} = 0. \quad (2.45)$$

Proof. Let constant a be the one from the condition (1.13). Then for any

$$0 \leq q < a/2 \quad (2.46)$$

let us define an auxiliary probability measure on S :

$$\mu_q(k) = m_q e^{q\psi(k)} \mu(k), \quad (2.47)$$

where the normalizing constant

$$m_q := \left(\sum_S e^{q\psi(k)} \mu(k) \right)^{-1} > 0.$$

Notice that $\mu_0(k) = \mu(k)$ for all $k \in S$, and m_q is continuous on $[0, a/2]$ with $m_0 = 1$. Fix $\varepsilon > 0$ and $0 < q < a/2$ arbitrarily and define an event

$$\mathcal{B}_n = \left\{ \left| \frac{\#\{1 \leq i \leq n : x_i = k\}}{n} - \mu(k) \right| \leq \varepsilon \mu_q(k), \quad k \in S \right\}. \quad (2.48)$$

Let $|\tau_n(x)|$ denote the number of vertices in $\tau_n(x)$. Then we easily derive taking into account the assumption (1.14) that

$$\begin{aligned} \mathbf{P} \left\{ C_1 \left(G^\vee(n, \kappa) \right) > n^{1/2} \right\} &\leq \mathbf{P} \left\{ \max_{1 \leq i \leq n} |\tau_n(x_i)| > n^{1/2} \mid \mathcal{B}_n \right\} + o(1) \\ &\leq n \sum_{k \in S} (\mu(k) + \varepsilon \mu_q(k)) \mathbf{P} \left\{ |\tau_n(k)| > n^{1/2} \mid \mathcal{B}_n \right\} + o(1) \end{aligned} \quad (2.49)$$

as $n \rightarrow \infty$.

To approximate the distribution of $|\tau_n(k)|$ we shall use the following branching processes. Let $B_{c,q}$ be a process defined similar to B_κ , but with the distribution of the offspring

$$Po(c\psi(x)\psi(y)\mu_q(y))$$

instead of $Po(c\psi(x)\psi(y)\mu(y))$. Notice, that $B_{c,0}$ is defined exactly as B_κ . We set

$$c^{cr}(q) = \left(\sum_S \psi^2(x) \mu_q(x) \right)^{-1}.$$

Clearly, $c^{cr}(q)$ is a continuous function of q on $[0, a/2]$ with $c^{cr}(0) = c^{cr}$.

Let further $\mathcal{X}^{c,q}(k)$ denote the total number of the particles (including the initial one) produced by the branching process $B_{c,q}$ starting with a single particle of type k .

Proposition 2.1. *For any $c < c^{cr}$ one can find $q > 0$ and $c < c' < \min\{c^{cr}(q), c^{cr}\}$ arbitrarily close to 0 and c , correspondingly, such that for all $k \geq 1$ and all large n*

$$\mathbf{P} \left\{ |\tau_n(k)| > n^{1/2} \mid \mathcal{B}_n \right\} \leq e^{b_1(\log n)^4} \mathbf{P} \left\{ \mathcal{X}^{c',q}(k) > n^{1/2} \right\}, \quad (2.50)$$

where b_1 is some positive constant independent of k and n .

Proof. Observe that at each step of the exploration algorithm which defines τ_n , the number of the type y offspring of a particle of type x has a binomial distribution $Bin(N'_y, p_{xy}(n))$ where N'_y is the number of the remaining vertices of type y .

We shall explore the following relation between the binomial and the Poisson distributions. Let $Y_{n,p} \in Bin(n, p)$ and $Z_\lambda \in Po(\lambda)$, where $0 < p < 1/4$ and $\lambda > 0$. Then for all $k \geq 0$

$$\mathbf{P}\{Y_{n,p} = k\} \leq (1 + \gamma p^2)^n \mathbf{P}\{Z_{n \frac{p}{1-p}} = k\}, \quad (2.51)$$

where γ is some positive constant (independent of n , k and p).

Notice that conditionally on \mathcal{B}_n we have

$$N'_y \leq \#\{1 \leq i \leq n : x_i = y\} \leq n(\mu(y) + \varepsilon\mu_q(y)) \quad (2.52)$$

for each $y \in S$. The last inequality implies that for any y such that

$$\#\{1 \leq i \leq n : x_i = y\} > 0$$

we have

$$n(\mu(y) + \varepsilon\mu_q(y)) \geq 1. \quad (2.53)$$

By the assumptions (1.13), (1.12) and (2.46) we have for all large y

$$\mu(y) + \varepsilon\mu_q(y) \leq e^{-a\psi(y)} + \varepsilon m_q e^{(q-a)\psi(y)} \leq b_2 e^{(q-a)A_1\psi_0(y)} \leq b_2 e^{-aA_1\psi_0(y)/2},$$

where $0 < b_2 < 1 + 2\varepsilon$ for all small $q > 0$. Combining this with (2.53) we obtain for all large n and y such that $\#\{1 \leq i \leq n : x_i = y\} > 0$

$$\frac{1}{n} \leq \mu(y) + \varepsilon\mu_q(y) \leq b_2 e^{-aA_1\psi_0(y)/2}.$$

This implies that conditionally on \mathcal{B}_n

$$\max_{x \in \{x_1, \dots, x_n\}} \psi_0(x) \leq A_3 \log n$$

for some constant A_3 , and thus conditionally on \mathcal{B}_n

$$p_{x_i x_j}(n) \leq c \frac{(A_3 \log n)^2}{n} \quad (2.54)$$

for all $x_i, x_j \in \mathbf{V}$. This and (2.52) together with the continuity of m_q allow us for any fixed positive ε_1 to choose ε and q in (2.48) so that conditionally on \mathcal{B}_n we get

$$\begin{aligned} N'_y \frac{p_{xy}(n)}{1 - p_{xy}(n)} &\leq (\mu(y) + \varepsilon\mu_q(y)) \frac{np_{xy}(n)}{1 - p_{xy}(n)} \\ &\leq (1 + \varepsilon_1)\mu_q(y)c\psi(x)\psi(y) =: \mu_q(y)c'\psi(x)\psi(y) \end{aligned} \quad (2.55)$$

for all large n . In other words, for any $q > 0$ and $c' > c$ arbitrarily close to 0 and c , respectively, and such that

$$c < c' < \min\{c^{cr}(q), c^{cr}\}, \quad (2.56)$$

bound (2.55) holds for all large n .

Now according to (2.51) and (2.55)

$$\begin{aligned} \mathbf{P}\{Y_{N'_y, p_{xy}(n)} \geq k\} &\leq (1 + \gamma p_{xy}(n)^2)^{N'_y} \mathbf{P}\{Z_{N'_y \frac{p_{xy}(n)}{1-p_{xy}(n)}} \geq k\} \\ &\leq (1 + \gamma p_{xy}(n)^2)^n \mathbf{P}\{Z_{\mu_q(y)c'\psi(x)\psi(y)} \geq k\}. \end{aligned} \quad (2.57)$$

Hence, if conditionally on \mathcal{B}_n at each (of at most n) step of the exploration algorithm which reveals $\tau_n(k)$, we replace the $\text{Bin}(N'_y, p_{xy}(n))$ variable with the $Po(\mu_q(y)c'\psi(x)\psi(y))$ one, we arrive at the following bound using branching process $B_{c',q}(k)$ and bound (2.54):

$$\mathbf{P}\{|\tau_n(k)| > n^{1/2} \mid \mathcal{B}_n\} \leq \left(1 + \gamma \left(c \frac{(A_3 \log n)^2}{n}\right)^2\right)^{n^2} \mathbf{P}\{\mathcal{X}^{c',q}(k) > n^{1/2}\}. \quad (2.58)$$

This implies statement (2.50) of the Proposition. \square

Substituting (2.50) into (2.49) we derive for any $q > 0$ and $c' > c$ that

$$\mathbf{P}\left\{C_1\left(G^\nu(n, \kappa)\right) > n^{1/2}\right\} \leq b_3 n e^{b_1(\log n)^4} \sum_{k \in S} \mu_q(k) \mathbf{P}\left\{\mathcal{X}^{c',q}(k) > n^{1/2}\right\} + o(1) \quad (2.59)$$

as $n \rightarrow \infty$, where b_3 is some positive constant. By the Markov's inequality

$$\mathbf{P}\{\mathcal{X}^{c',q}(k) > n^{1/2}\} \leq z^{-n^{1/2}} \mathbf{E} z^{\mathcal{X}^{c',q}(k)} \quad (2.60)$$

for all $z \geq 1$. This bound together with (2.59) yield

$$\mathbf{P}\left\{C_1\left(G^\nu(n, \kappa)\right) > n^{1/2}\right\} \leq b_3 n e^{b_1(\log n)^4} z^{-n^{1/2}} \sum_{k \in S} \mu_q(k) \mathbf{E} z^{\mathcal{X}^{c',q}(k)} + o(1). \quad (2.61)$$

We are left to show that for some $z > 1$

$$\sum_{k \in S} \mu_q(k) \mathbf{E} z^{\mathcal{X}^{c',q}(k)} < \infty. \quad (2.62)$$

Note that μ_q and ψ satisfy the conditions of Lemma 2.1 for any $0 \leq q < a/2$, and moreover $c' < c^{cr}(q)$. Hence, by Lemma 2.1 there exists $z_0 > 1$ such that

$$\sum_{k \in S} \psi(k) \mu_q(k) \mathbf{E} z_0^{\mathcal{X}^{c',q}(k)} < \infty,$$

which clearly implies (2.62), and the statement of Lemma 2.2 follows by (2.61) where we set $z = z_0 > 1$. \square

Now we are ready to complete the proof of (2.44), following almost the same arguments as in the proof of the previous lemma. Denote

$$\mathcal{B}'_n := \mathcal{B}_n \cap \left(C_1 \left(G^\nu(n, \kappa) \right) \leq n^{1/2} \right).$$

According to assumption (1.14) and Lemma 2.2 we have

$$\mathbf{P} \{ \mathcal{B}'_n \} = 1 - o(1)$$

as $n \rightarrow \infty$. This allows us to derive similar to (2.49) for any ω

$$\begin{aligned} \mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) > \omega \right\} &\leq \mathbf{P} \left\{ \max_{1 \leq i \leq n} |\tau_n(x_i)| > \omega \mid \mathcal{B}'_n \right\} + o(1) \\ &\leq n \sum_{k \in S} (\mu(k) + \varepsilon \mu_q(k)) \mathbf{P} \{ |\tau_n(k)| > \omega \mid \mathcal{B}'_n \} + o(1) \end{aligned} \quad (2.63)$$

as $n \rightarrow \infty$. Repeating the same argument which led to (2.58), we get the following bound using the introduced branching process:

$$\mathbf{P} \{ |\tau_n(k)| > \omega \mid \mathcal{B}'_n \} \leq \left(1 + \gamma \left(c \frac{(A_3 \log n)^2}{n} \right)^2 \right)^{b_1 n \sqrt{n}} \mathbf{P} \left\{ \mathcal{X}^{c', q}(k) > \omega \right\}$$

as $n \rightarrow \infty$, where we take into account that we can perform at most \sqrt{n} steps of exploration (the maximal possible number of macro-vertices in any connected component conditioned on \mathcal{B}'_n). Notice also that we can choose here c' and q arbitrarily close to c and 0, correspondingly, and so that condition (2.56) is fulfilled. The last bound implies

$$\mathbf{P} \{ |\tau_n(k)| > \omega \mid \mathcal{B}'_n \} \leq (1 + o(1)) \mathbf{P} \left\{ \mathcal{X}^{c', q}(k) > \omega \right\} \quad (2.64)$$

as $n \rightarrow \infty$. Substituting (2.64) into (2.63) we derive

$$\mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) > \omega \right\} \leq bn \sum_{k \in S} \mu_q(k) \mathbf{P} \left\{ \mathcal{X}^{c', q}(k) > \omega \right\} + o(1) \quad (2.65)$$

as $n \rightarrow \infty$, where b is some positive constant. Then similar to (2.61) we derive from (2.65)

$$\mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) > \omega \right\} \leq bn z^{-\omega} \sum_{k \in S} \mu_q(k) \mathbf{E} z^{\mathcal{X}^{c', q}(k)} + o(1). \quad (2.66)$$

Since ψ and probability μ_q satisfy the conditions of Lemma 2.1, for all values c' and q for which (2.56) holds, we have by this lemma

$$r_q(c') := \sup\{z \geq 1 : \sum_{x \in S} \psi(x) \mathbf{E} z^{\chi^{c', q}(x)} \mu_q(x) < \infty\} > 1,$$

and therefore for all $1 < z < r_q(c')$

$$\sum_{k \in S} \mu_q(k) \mathbf{E} z^{\chi^{c', q}(k)} < \infty. \quad (2.67)$$

Note that $r_q(c')$ is a continuous function of q and c' : this can be explicitly derived from formula (2.43) where X is replaced by a random variable X_q with a probability function μ_q . Moreover, $r_0(c) = r(c)$. Hence, for any $\delta > 0$ we can choose a small $\delta' > 0$ and (q, c') close to $(0, c)$ so that (2.67) holds with

$$z = r_q(c') - \delta' > 1, \quad (2.68)$$

and also

$$\left(\frac{1}{\log r(c)} + \delta \right) \log(r_q(c') - \delta') > 1. \quad (2.69)$$

Now setting $\omega = \left(\frac{1}{\log r(c)} + \delta \right) \log n$ and $z = r_q(c') - \delta'$ in (2.66) we derive with a help of (2.67)

$$\begin{aligned} \mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) > \left(\frac{1}{\log r(c)} + \delta \right) \log n \right\} &\leq b_3 n z^{-\omega} + o(1) \\ &= b_3 n \exp\{-\log(r_q(c') - \delta') \left(\frac{1}{\log r(c)} + \delta \right) \log n\} + o(1) \end{aligned} \quad (2.70)$$

where b_3 is some finite positive constant. This together with (2.69) clearly implies statement (2.44). \square

2.3.2 The upper bound.

Here we show that for any $\delta > 0$

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) < \left(\frac{1}{\log r(c)} - \delta \right) \log n \right\} = 0. \quad (2.71)$$

Given a random graph $G^\mathcal{V}(n, \kappa)$ let T be the number of its connected components, and denote W_i , $i = 1, \dots, T$, the corresponding sets of the vertices in these components, ordered arbitrarily. Let us fix $\varepsilon > 0$ arbitrarily and for each $n \geq 1$ introduce an event

$$\mathcal{A}(n) = \left\{ \max_{1 \leq i \leq T} |W_i| \leq \left(\frac{1}{\log r(c)} + \varepsilon \right) \log n \right\} \cap \mathcal{B}_n$$

with \mathcal{B}_n defined in (2.48). According to (2.44) and the assumption (1.14)

$$\mathbf{P} \{ \mathcal{A}(n) \} \rightarrow 1 \quad (2.72)$$

as $n \rightarrow \infty$. Let

$$\mathbf{P}_{\mathcal{A}(n)}(\cdot) = \mathbf{P} \{ \cdot \mid \mathcal{A}(n) \}$$

denote the conditional probability. Denote further

$$\omega = \left(\frac{1}{r(c)} - \delta \right) \log n, \quad \omega_1 = \left(\frac{1}{r(c)} + \varepsilon \right) \log n, \quad (2.73)$$

$$N = N(n) = \frac{n}{\omega_1^2}. \quad (2.74)$$

Clearly,

$$\mathbf{P}_{\mathcal{A}(n)} \{ T \geq [N] + 1 \} = 1. \quad (2.75)$$

We shall reveal recursively $[N] + 1$ connected components in the graph $G^\mathcal{V}(n, \kappa)$ in the following way. Let V_1 be a random vertex uniformly distributed on \mathbf{V} . Set $L_1 = \tau_n(V_1)$ to be the set of the vertices in the connected component containing vertex V_1 .

Further for any $U \subset \mathbf{V}$ let $\tau_n^U(v)$ denote a set of vertices of the tree constructed in the same way as $\tau_n(v)$ but on the set of vertices $\mathbf{V} \setminus U$ instead of \mathbf{V} . In particular, with this notation $\tau_n^\emptyset(v) = \tau_n(v)$.

Given constructed components L_1, \dots, L_k for $1 \leq k \leq [N]$, let V_{k+1} be a vertex uniformly distributed on $V \setminus \cup_{i=1}^k L_i$, and set $L_{k+1} = \tau_n^{\cup_{i=1}^k L_i}(V_{k+1})$. Then according to (2.75) and (2.72) we have

$$\mathbf{P} \left\{ C_1(G^\mathcal{V}(n, \kappa)) < \left(\frac{1}{\log r(c)} - \delta \right) \log n \right\} \leq \mathbf{P}_{\mathcal{A}(n)} \left\{ \max_{1 \leq i \leq [N]+1} |L_i| < \omega \right\} + o(1) \quad (2.76)$$

as $n \rightarrow \infty$.

Let $x_0 \in S$ be such that $\psi(x_0) = \min_{x \in S} \psi(x)$. Then a vertex of type x_0 has among all different types $x \in S$ the smallest probabilities of the incident edges, which are $c\psi(x_0)\psi(y)/n$, $y \in S$. This implies that for any $U \subset \mathbf{V}$ the size of $\tau_n^U(x_0)$ is stochastically dominated by

$|\tau_n^U(x)|$ for any $x \in S$. Notice also that if $U \subset U'$ then $|\tau_n^{U'}(x)|$ is stochastically dominated by $|\tau_n^U(x)|$ for any $x \in S$. This allows us to derive the following bound

$$\mathbf{P}_{\mathcal{A}(n)} \left\{ \max_{1 \leq i \leq [N]+1} |L_i| < \omega \right\} \leq \left(\max_{U \subset \mathbf{V}: |U| \leq N\omega_1} \mathbf{P}_{\mathcal{A}(n)} \left\{ |\tau_n^U(x_0)| < \omega \right\} \right)^N. \quad (2.77)$$

To approximate the distribution of $|\tau_n^U(x)|$ we introduce another branching process which will be stochastically dominated by B_κ . First define for any value $D \in S$ another auxiliary probability measure $\hat{\mu}_D$

$$\hat{\mu}_D(y) = \begin{cases} M_D^{-1} \mu(y), & \text{if } y \leq D, \\ 0, & \text{otherwise,} \end{cases} \quad (2.78)$$

where $M_D := \sum_{y \leq D} \mu(y)$ is a normalizing constant. Then for any positive c and D let $\hat{B}_{c,D}$ be a process defined similar to B_κ , but with the distribution of the offspring

$$Po(c\psi(x)\psi(y)\hat{\mu}_D(y)) \quad (2.79)$$

instead of $Po(c\psi(x)\psi(y)\mu(y))$. Notice, that $\hat{B}_{c,\infty}$ is defined exactly as B_κ . Let $\hat{\mathcal{X}}^{c,D}(x)$ denote the total number of the particles (including the initial one) produced by the branching process $\hat{B}_{c,D}$ starting with a single particle of type x .

Lemma 2.3. *For any $c' < c$ there exists finite D such that*

$$\mathbf{P}_{\mathcal{A}(n)} \left\{ |\tau_n^U(x_0)| < \omega \right\} \leq \left(1 + b \frac{\log^4 n}{n^2} \right)^{n\omega_1} \mathbf{P} \left\{ \hat{\mathcal{X}}^{c',D}(x_0) < \omega \right\} \quad (2.80)$$

for all large n uniformly in $U \subset \mathbf{V}$ with $|U| \leq N\omega_1 = \left(\frac{1}{r(c)} + \varepsilon \right)^{-1} n / \log n$, where $b = b(c)$ is some positive constant independent of c' and D .

Proof. At each step of the exploration algorithm which defines $\tau_n^U(x_0)$, the number of the type y offspring of a particle of type x has a binomial distribution $Bin(N'_y, p_{xy}(n))$ where N'_y is the number of remaining vertices of type y . We shall first find a lower bound for N'_y .

According to assumption (1.1) for any $D \in S$ and $\varepsilon_1 > 0$ there exists $n(D, \varepsilon_1)$ such that

$$N_y := \#\{x_i \in \mathbf{V} : x_i = y\} \geq (1 - \varepsilon_1) \mu(y) n \quad (2.81)$$

for all $y < D$ and $n \geq n(D, \varepsilon_1)$. Note that conditionally on $\mathcal{A}(n)$ the number of vertices in $\tau_n^U(x)$ is at most ω_1 , and by deleting an arbitrary set U with $|U| \leq N\omega_1$ from \mathbf{V} , we may delete at most $N\omega_1$ vertices of type y . Hence, at any step of the exploration algorithm which

defines $\tau_n^U(x)$, the number N'_y of the remaining vertices of type y , is bounded from below as follows

$$N'_y \geq N_y - \omega_1 - N\omega_1,$$

and thus according to (2.81)

$$N'_y \geq n(1 - \varepsilon_1)\mu(y) - \omega_1 - N\omega_1$$

for all $y < D$ and $n \geq n(D, \varepsilon_1)$. Taking into account definitions (2.74) and (2.73) we derive from here that for any $\varepsilon_1 > 0$ and $D > 0$ there exists $n(D, \varepsilon_1)$ such that

$$N'_y \geq (1 - \varepsilon_1)\mu(y)n$$

for all $y \leq D$ and $n \geq n(D, \varepsilon_1)$. This implies that conditionally on $\mathcal{A}(n)$ at any step of the exploration algorithm we have

$$N'_y \frac{p_{xy}(n)}{1 - p_{xy}(n)} \geq \mu(y)(1 - \varepsilon_1)c\psi(x)\psi(y) \quad (2.82)$$

for any fixed ε_1, D , and all $n \geq n(D, \varepsilon_1)$ and $y \leq D$. Now with a help of (2.78) we rewrite (2.82) as follows:

$$N'_y \frac{p_{xy}(n)}{1 - p_{xy}(n)} \geq \hat{\mu}_D(y)M_D(1 - \varepsilon_1)c\psi(x)\psi(y) =: \hat{\mu}_D(y)c'\psi(x)\psi(y) \quad (2.83)$$

for any fixed ε_1, D , and all $n \geq n(D, \varepsilon)$ and $x, y \in S$, where

$$c' = M_D(1 - \varepsilon_1)c.$$

Recall that $\lim_{D \rightarrow \infty} M_D \uparrow 1$. Therefore choosing appropriately constants D and ε_1 we can make c' arbitrarily close to c . Using again relation (2.51) between the Poisson and the binomial distributions, and taking into account (2.83), we derive for all k

$$\mathbf{P}\{Y_{N'_y, p_{xy}(n)} \leq k\} \leq (1 + \gamma p_{xy}^2(n))^{N'_y} \mathbf{P}\{Z_{N'_y \frac{p_{xy}(n)}{1 - p_{xy}(n)}} \leq k\} \quad (2.84)$$

$$\leq (1 + \gamma p_{xy}^2(n))^n \mathbf{P}\{Z_{\hat{\mu}_D(y)c'\psi(x)\psi(y)} \leq k\}.$$

This implies that if conditionally on \mathcal{A}_n , at each of at most ω_1 steps of the exploration algorithm which reveals $\tau_n^U(x_0)$, we replace the $\text{Bin}(N'_y, p_{xy}(n))$ variable with the

$$\text{Po}(\hat{\mu}_D(y)c'\psi(x)\psi(y))$$

one, we arrive at the following bound using the branching process $\hat{B}_{c,D}$ and bound (2.54)):

$$\mathbf{P}_{\mathcal{A}(n)} \left\{ |\tau_n^U(x_0)| < \omega \right\} \leq \left(1 + \gamma c^2 A_3^4 \frac{\log^4 n}{n^2} \right)^{n\omega_1} \mathbf{P} \left\{ \hat{\mathcal{X}}^{c',D}(x_0) < \omega \right\} \quad (2.85)$$

which holds for all large n . This implies the statement of Lemma 2.3. \square

Lemma 2.3 together with (2.77) implies that for all large n

$$\mathbf{P}_{\mathcal{A}(n)} \left\{ \max_{1 \leq i \leq [N]+1} |L_i| < \omega \right\} \leq \left(\left(1 + b \frac{\log^4 n}{n^2} \right)^{n\omega_1} \mathbf{P} \left\{ \hat{\mathcal{X}}^{c',D}(x_0) < \omega \right\} \right)^N.$$

Substituting this into (2.76) we derive

$$\begin{aligned} \mathbf{P} \left\{ C_1 \left(G^{\mathcal{V}}(n, \kappa) \right) < \omega \right\} &\leq \left(\left(1 + b \frac{\log^4 n}{n^2} \right)^{n\omega_1} \mathbf{P} \left\{ \hat{\mathcal{X}}^{c',D}(x_0) < \omega \right\} \right)^N + o(1) \\ &= \left(1 + b \frac{\log^4 n}{n^2} \right)^{n^2/\omega_1} \mathbf{P} \left\{ \hat{\mathcal{X}}^{c',D}(x_0) < \omega \right\}^{n/\omega_1^2} + o(1) \\ &\leq e^{b_1 \log^3 n} \left(1 - \mathbf{P} \left\{ \hat{\mathcal{X}}^{c',D}(x_0) \geq \omega \right\} \right)^{n/\omega_1^2} + o(1) \end{aligned} \quad (2.86)$$

as $n \rightarrow \infty$, where $b_1 = b_1(c)$ is a positive constant which depends only on c .

Since $\hat{\mu}_D$ and ψ obviously satisfy the conditions of Lemma 2.1 we get by (2.34) and (2.32) that if

$$c' < \hat{c}^{cr}(D) := \left(\sum_S \psi^2(x) \hat{\mu}_D(x) \right)^{-1},$$

then

$$\hat{r}(c', D) := \sup \{ z \geq 1 : \mathbf{E} z^{\hat{\mathcal{X}}^{c',D}(x_0)} < \infty \} > 1,$$

and $\hat{r}(c', D)$ can be derived from (1.21) where X is replaced by a random variable on S with a distribution $\hat{\mu}_D$. It is clear that

$$\lim_{D \rightarrow \infty} \hat{c}^{cr}(D) = c^{cr},$$

$\hat{r}(c, D)$ is continuous in c , and

$$\lim_{D \rightarrow \infty} \hat{r}(c, D) = r(c).$$

Hence we can find for any given $\delta_1 > 0$ a large constant D and $c' < c$ sufficiently close to c , such that

$$\hat{r}(c', D) < r(c) + \delta_1/2.$$

Then it follows from the definition of $\hat{r}(c', D)$ that for some positive constant $A = A(\delta_1) < \infty$ and any positive ω

$$\mathbf{P} \left\{ \hat{\mathcal{X}}^{c', D}(x_0) > \omega \right\} \geq A(\hat{r}(c', D) + \delta_1/2)^{-\omega} \geq A(r(c) + \delta_1)^{-\omega}. \quad (2.87)$$

This allows us to derive from (2.86) that for any $\delta > 0$, $\delta_1 > 0$ and some positive A

$$\mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) < \omega \right\} \quad (2.88)$$

$$\leq e^{b_1 \log^3 n} \left(1 - A(r(c) + \delta_1)^{-\left(\frac{1}{\log r(c)} - \delta\right) \log n} \right)^{\alpha^2 n / \log^2 n} + o(1)$$

where $\alpha = \left(\frac{1}{\log r(c)} + \varepsilon \right)^{-1}$. Now for any $\delta > 0$ we choose a positive δ_1 so that

$$\gamma_1 := \left(\frac{1}{\log r(c)} - \delta \right) \log(r(c) + \delta_1) < 1.$$

Then (2.88) becomes

$$\mathbf{P} \left\{ C_1 \left(G^\nu(n, \kappa) \right) < \left(\frac{1}{\log r(c)} - \delta \right) \log n \right\} \leq e^{b_1 \log^3 n} \left(1 - \frac{A}{n^{\gamma_1}} \right)^{\alpha^2 n / \log^2 n} + o(1), \quad (2.89)$$

where the right-hand side goes to zero when $n \rightarrow \infty$. This completes the proof of (2.71), which together with (2.44) yields the assertion of Theorem 1.1. \square

2.4 Proof of Theorem 1.3.

The proof of Theorem 1.3 almost exactly repeats the proof of Theorem 1.1. The only difference is that a random variable $|\tau_n(x)|$ used in the proof of Theorem 1.1 should be replaced by

$$\Psi_n(x) := \sum_{v \in \tau_n(x)} \psi(v),$$

while $\mathcal{X}^{c, q}$ and $\hat{\mathcal{X}}^{c, D}$ should be replaced by $\Phi^{c, q}$ and $\hat{\Phi}^{c, D}$ which denote the activity of the total progeny of the branching processes $B_{c, q}$ and $\hat{B}^{c, D}$, correspondingly (see definition of Φ and (1.22)). Then due to the results (2.35) and (2.33) on α from Lemma 2.1 the proof of Theorem 1.3 follows exactly the same lines as the proof of Theorem 1.1. \square

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